Abstract - This paper describes a decomposition technique to be used in conjunction with algorithms for the solution of small-signal stability problems of large power systems [1]. This technique decomposes the complete system into one internal system and several interconnected external systems and allows efficient use of parallel computation. This technique is also valuable in frequency response studies, on a uniprocessor computer, when the applied disturbance and monitored outputs are all located within the internal system.

1. INTRODUCTION

Reference 1 presents efficient algorithms for the solution of small-signal stability problems of large power systems. These problems include the calculation of dominant eigenvalues, frequency response plots of transfer functions between any two variables in the system and step response solutions by numerical integration.

This paper describes a technique to be used in connection with these algorithms which decomposes the complete electric system into one internal system and several external systems. The formulation developed is general and each external system can be connected to any other external system and to the internal system through various boundary buses. There is no need for system stability considerations to define the external and internal systems, since the decomposition technique is just used as a mathematical tool for efficient solution.

The decomposition technique finds application in three instances:
1. To solve large problems on uniprocessor computers with reduced core capacity. The decomposition technique is then used to break apart the complete power system into smaller subsystems which are individually solved.
2. To save computation time on a uniprocessor computer when using the frequency response algorithm described in [1]. In this case it is assumed that the applied disturbance and monitored outputs are all located within the internal system [1].
3. To speed up the solution of power system small-signal stability problems through use of parallel computation [2].

The power of the algorithms proposed in [1] rests in the fact that the electric network equations are expanded in their unwound form. This fact also accounts for the efficiency of the decomposition technique, which is similar and equally as flexible as those proposed in [4,5] for the solution of steady-state power system problems.

II. MATHEMATICAL FORMULATION

Reference 1 presents efficient algorithms for the solution of small-signal stability problems of large power systems (see Appendix). All these algorithms involve the solution of an equation of the general form:

$$\mathbf{J}_d \mathbf{x} = \mathbf{b}$$

where $\mathbf{J}_d$ corresponds to the matrix block $\mathbf{J}_d$ of Figure 1 with some minor terms added to its diagonal. The characteristics of the Jacobian matrix shown in Figure 1 are described in the Appendix.

![Figure 1: The Power System Jacobian Matrix](Image)

Consider an interconnected system which is to be decomposed into two subsystems. By reordering the variables of equation (1) one obtains:

...
where subscripts "G", "I" and "B" refer to variables or matrix blocks associated with the external system, internal system and boundary buses, respectively. All variables in (2) are incremental values, but the symbol Δ is omitted for simplicity.

The correspondence between blocks of (1) and (2) is as follows:

\[ J_A = \begin{bmatrix} A_1 & A_2 & \vdots & A_N \\ A_2^T & B_1 & \vdots & B_N \\ \vdots & \vdots & \ddots & \vdots \\ A_N^T & B_N & \vdots & C_1 & \end{bmatrix} \]

\[ J_C = \begin{bmatrix} C_1 & C_2 & \vdots & C_N \\ C_2^T & C_1 & \vdots & C_N \\ \vdots & \vdots & \ddots & \vdots \\ C_N^T & C_N & \vdots & C_1 \\ \end{bmatrix} \]

Elimination of the external system variables \( (\Delta G, \Delta V) \) and independent vector \( \Delta B \) is then performed, yielding:

\[ \begin{bmatrix} A_1 & A_2 & \vdots & A_N \\ A_2^T & B_1 & \vdots & B_N \\ \vdots & \vdots & \ddots & \vdots \\ A_N^T & B_N & \vdots & C_1 \end{bmatrix} \begin{bmatrix} \Delta V \end{bmatrix} = \begin{bmatrix} \Delta \end{bmatrix} \]

where:

\[ \begin{align*}
\Delta V &= V_{GB} - V_{EB} - V_{VGB} (Y_{VGB})^{-1} V_{EB} \\
V_{GB} &= V_{EB} - C_1 (A_1)^{-1} A_2^T \\
V_{EB} &= V_{EB} - C_1 (A_1)^{-1} A_2^T \\
A_2^T &= C_1 (A_1)^{-1} A_2^T \\
A_2^T &= C_1 (A_1)^{-1} A_2^T
\end{align*} \]

Subvectors \( \Delta \), \( \Delta B \) and \( \Delta V \) can be solved by using either equation (2) or (3). The solution vector \( \Delta \) formed by subvectors \( \Delta V \), \( \Delta B \), \( \Delta Y \), and \( \Delta Y_{VGB} \) can also be obtained by decomposing (3) into two sets of equations:

The matrices of equations (2), (3), (4), (5) and (7) are highly sparse (see Appendix) and the decomposition technique is computationally efficient with the use of sparsity-oriented routines for Gaussian Elimination.

Note that the equations shown in this section can easily be generalized for the case of multiple external systems, but this is here omitted for brevity. A similar formulation, devoted to steady-state power system analysis, can be found in [5].

It should be emphasized that the decomposition technique does not introduce any potential numerical instability problems into the basic algorithms reported in [11]. The use of equations (2) or (4a) and (4b) lead to the same numerical values for the solution vector. The difference in round-off errors between the two cases is insignificant to any engineering application. Therefore there is no degradation in the convergence rate of the algorithms presented in [1] when using the proposed decomposition technique.

III. CONSIDERATIONS ON THE DECOMPOSITION PRINCIPLE

Consider the electric system shown in Figure 2a which comprises an internal system and three interconnected external systems. Buses 61, 62 and 63 define the external system boundary while buses 81, 82 and 83 represent the boundaries between external systems. In order to apply the decomposition methodology described in Section II it is necessary to define an Extended Internal System comprising the original internal system plus all the boundary buses in the complete system. Connectivity of the Extended Internal System comes through the elements or "lines" generated by Gaussian Elimination of all external system variables, as shown in Figure 2b. The equivalent shorts and injected "currents" at the boundary buses, generated
The Jacobi matrix of each external system is formed with its network equations optimally ordered per Timney-2 scheme [6]. In the Jacobi matrix formation of the Extended Internal System, the Timney-2 ordering scheme is carried out considering the external system equivalents plus the Extended Internal System (see Figure 2b).

**Figure 2a**: Diagram of Interconnected System

**Figure 2b**: Extended Internal System plus External Equivalents (The equivalent buses at the boxes are omitted)

It may be convenient to retain selected buses in order to preserve the stability of the external system equivalents [6]. The decomposition technique can consider this by simply including the retained external buses in the boundary bus partitioning of the Extended Internal System. It is interesting to note that the Extended Internal System can be comprised of only a set of non-connected buses and does not need to have components with a dynamic representation. Let us again consider the system of Figure 2 and define the depicted internal system as the fourth external system and the set of six boundary buses as the new Extended Internal System. The solution of the Extended Internal System variables (stage 4 of the computational scheme presented in Figure 3) involves the factorization of matrix $B_{ij}$, which is here associated with a system of six buses and 11 equivalent lines (see Section III).

**Figure 3**: Suggested Implementation on a Multiprocessor Network

The decomposition technique presented in this paper allows efficient implementation on a parallel multiprocessor, in which every CPU would specifically solve a designated subsystem. Thus it is convenient to break apart the interconnected system into a number of subsystems of approximately the same size. A number of considerations need to be made in order to define these subsystems, since the decomposition scheme is just used as a mathematical tool for efficient parallel solution. The definition of the internal system and the Extended Internal System and the blocks identified as $B_{ij}$ and $B_{ij}$ represent different processors solving the $i$-th External System and the Extended Internal System respectively. It is seen that this parallel computation scheme has a bottleneck associated with the solution for the Extended Internal System variables. It is important to select an adequate set of boundary buses so as to preserve sparsity in the Extended Internal System matrix, and also to keep it as small as possible.

The system decomposition technique can be efficiently implemented on a simple multiprocessor structure: independent processors connected on a single bus. There will not exist problems of high communication overhead as most of the computational work is independently done by the various processors in parallel. As seen in Figure 3, there is no data transmission between processors which are in parallel.

**Speed-Up Obtained With The Parallel Computation Scheme**

Some figures are now given on the speed up obtained by using the decomposition technique on a parallel computation scheme. The major part of the computational work in the implementation shown in Figure 3 is devoted to the execution of steps 3, 4, 5. The computational effort for sparse-oriented factorization and solution will be considered to vary linearly with matrix size. Accordingly, on ideal conditions the implementation shown in Figure 3 is expected to require a time step $T_{max}$ proportional to the time step of steps 3, 4, 5. In other words, step 3 accounts for the most part of the total time step. Then it is assumed that the time step is not to be of the same size and it is the number of subsystems in which the interconnected system was decomposed. The constraint is that the maximum time step needed for sequential solution of the complete interconnected system on a single processor. For the case of 50 subsystems of equal size, it is assumed that the number of independent processors, the maximum speed up obtained from the method described parallel computation scheme would be of the order of 20 times.

The task of decomposing the interconnected system into a few subsystems can easily be done according to geographical units such as the individual utilities, "areas" and "subareas." However, the automatic decomposition is not always possible. For maximum efficiency of the described parallel computation scheme would require very clever ordering algorithms of the block-bordered diagonal form (BBDF) [4, 5].

**IV. RESULTS**

**Frequency Response Calculations**

The decomposition technique applied to the frequency response algorithm [1] on a multiprocessor computer, can significantly reduce computational work. It is recommended for use in engineering studies in which the applied disturbances and more precisely the all injected within the internal system. Subvector $B_{ij}$ of $C_{ij}$ does not exist in the application of the domain $C_{ij}$ of the multiprocessor system. The external system frequency response components are then normalized and stored on disk files for every discrete value of applied frequency within the range of interest. A typical study case could have few generating plants as the internal system and the whole interconnected system as the external system. In this
The simulation program is still at prototype level; the code needs to be further optimized and it only allows a system of two boundary boxes per external system. As a consequence, only a low number of subsystems of very different sizes could be generated (see Table 1). This precluded a correct evaluation of the speed up which can be obtained by the parallel computation scheme.

The eigenvalue pair associated with an inter-area mode of oscillation, in which the whole Southeast System swings against the South System, is $\lambda = -0.113 \pm 3.540$. The small-signal stability program of [1] was used to find this eigenvalue by the Implicit Vector Iteration (IVI) algorithm starting from an estimate $\hat{\lambda} = 0 + j \lambda$. The IIV algorithm converged to a tolerance of $10^{-6}$ in 11 iterations in 51 seconds of CPU on a VAX-11/780 computer. The simulation program was run for four cases in which the system was divided into 2, 4, 3, and 5 subsystems. In all cases the eigenvalue results were the same (up to 7 significant figures) as that obtained with the original non-parallelized system. The simulation program required 100 seconds of CPU to run the case with 5 subsystems, due to the schedulable disk operations during the convergence process and the less efficient code.

The natural advantage of the parallel computation simulation program is that new subproblems "times larger than the memory capacity of the uniprocessor computer ("one is the number of subsystems in which the system is divided})."

V. CONCLUSIONS

The decomposition technique is very effective in saving computation on a uniprocessor computer when conducting frequency response studies on multimachine systems.

The suggested decomposition technique applied to the algorithms of [1] made possible the parallel computation of small-signal instability problems. In general, the larger the number of subsystems in which the system is divided, the faster the desired calculation.

The parallel computation of eigenvalues, via AEPS and Implicit Vector Iteration algorithms [1], was simulated on a uniprocessor computer. Results were obtained for a multimachine interconnected system which was divided into five subsystems according to natural geographical units, such as "west" and "southeast." Further partitioning of the system was not attempted due to present limitations of the prototype version of the program. This precluded an evaluation of the speed up which can be obtained by parallel computation.

It was observed that there is no degradation in the results or convergence rate of the algorithms presented in [1] when using the decomposition technique.

Efficient algorithms are needed to automatically decompose a given power system into many subsystems and to the most convenient form for parallel computation.

Subtractive [9] also could have been used in connection with the algorithms of [1], but the decomposition technique described leads to simpler computer implementation.
(1) Complete system is broken apart into various subsystems.

(2) Form Jacobian matrix for all subsystems.

(3) Compute external system equivalents and the injected currents at the boundary buses for the given perturbation frequency \( \omega_p \).

(4) Compute variables for the internal system after incorporating all external systems contributions. 

With knowledge of the boundary bus conditions, compute the variables for the various external systems.

(5) The solution for the complete system has been obtained, and the new perturbation frequency \( \omega_p^{(k+1)} \) can be calculated.

(6) At convergence, the desired eigenvalue is \( \lambda = \omega_p^{(k)} \), where \( k \) is the iteration number.

(7) The blocks identified as \( ES_i \) and IS represent different processors solving the "i-th" external system and the Extended Internal System respectively.
The Jacobian matrix is formed by First entering the blocks of equations for every system generator. After this the induction motor blocks are built and are followed by the static VAR compensator blocks. The network equations come last in the Jacobian matrix formation. Submatrix Jp, shown in Figure 1, is practically equal to the nodal admittance matrix of the network, expanded into its real and imaginary parts. The only difference lies in the fact that the (2x2) diagonal blocks of Jp corresponding to the nodes which contain linear, non-linear or induction motor loads, have extra partial derivative terms added to them.

The power system Jacobian matrix is extremely sparse and of very high order, and the full use of sparsity techniques becomes imperative. As seen from Figure 1, there will be practically no fill-in during the LU factorization of the submatrix Jp and therefore the ordering of its equations is not critical.[5] The network equations of submatrix Jp are optimally ordered per Time-2 scheme which selects the node for elimination at each stage that introduces the fewest non-zero elements [4].

APPENDIX

EFFICIENT ALGORITHMS FOR THE SOLUTION OF EIGEN-LIKE LOCAL-STABILITY PROBLEMS

The package of computer programs developed in the work reported in [1] contains efficient algorithms for the calculation of:
1. Exact eigenvalue closest to a specified point in a complex plane, and associated eigenvector (implicit inverse iteration algorithm).
2. Exact eigenvalue associated with the dominant mode of oscillation of the system (LAPACK algorithm) [7, 8].
3. Frequency response plots of transfer functions between any two specified variables in the system.
4. Step response results obtained via implicit trapezoidal integration formula. This algorithm was not described in [11] for being of straightforward implementation.

All the aforementioned functions of the program package make use of the Jacobian matrix of the differential-algebraic set of equations for the electric system, calculated for a system operating point. All the pertinent formulation has been described in [1]. The Jacobian matrix, whose structure is depicted in Figure 1, contains for various models of synchronous generators and associated controllers, induction motors, non-linear loads of different characteristics and static VAR compensators. Special routines also allow the user to specify power system controllers without restriction on their order or topology.